RELATIONS BETWEEN

$\otimes^{k}\mathcal{B} = \underbrace{\mathcal{B}\otimes \cdots \otimes \mathcal{B}}_{k}$ AND RSK.

$\mathcal{B}$: STANDARD $GL(n)$ CRYSTAL.

$$\boxed{1} \xrightarrow{1} \boxed{2} \xrightarrow{2} \boxed{3} \rightarrow \cdots \xrightarrow{n-1} \boxed{n}$$

$GL(n)$ CRYSTAL

$x \xrightarrow{a} y$ MEANS

$f_a(x) = y$

$e_a(y) = x$.

IF λ IS ANY PARTITION OF k

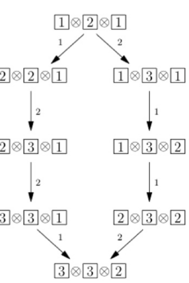
WE CAN FIND (SOMETIMES MORE THAN ONE)

COPY OF $\mathcal{B}_\lambda$ IN $\otimes^{k}\mathcal{B}$.

EXAMPLE: CRYSTALS OF ROWS

$$\mathcal{B}_{(k)} = \{ \underbrace{\boxed{\;\;|\;\;|\;\; \cdots \;\;|\;\;}}_{k \text{ BOXES}} \}$$

$(k) = \lambda$ ROWS WEAKLY INCREASING



EMBED AS FOLLOWS

$$\boxed{a_1}\boxed{a_2}\cdots\boxed{a_n} \rightsquigarrow \boxed{a_1}\otimes\boxed{a_2}\otimes\cdots\otimes\boxed{a_n}.$$

EXAMPLE: CRYSTALS OF COLUMNS.

$$\mathcal{B}_{(1^n)} = \left\{ \left.\begin{array}{|c|} \hline a_1 \\ \hline \vdots \\ \hline a_n \\ \hline \end{array}\right\}^n \;\middle|\; a_1 < a_2 < \cdots < a_n \right\}$$

$$(1^n) = (\underbrace{1, \cdots, 1}_{n \text{ TIMES.}})$$

EMBEDS AS $\boxed{a_n}\otimes\cdots\otimes\boxed{a_1}$.

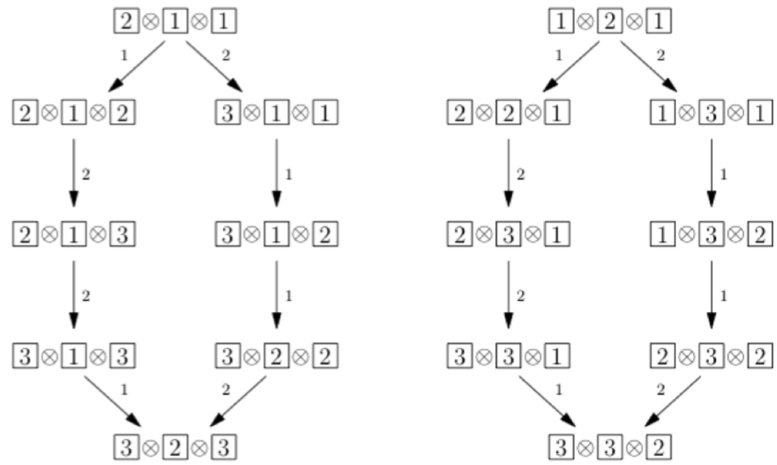
OBSERVE THIS IS DISJOINT FROM

$\mathcal{B}_{(n)}$

EXAMPLE: THERE ARE TWO COPIES OF

$$\mathcal{B}_{(2,1)} \text{ IN } \mathcal{B}\otimes\mathcal{B}\otimes\mathcal{B}.$$

CASE $n = 3$ BELOW (FROM BOOK)



FOR GL(3) $n = 3$ $\boxed{a} \otimes \boxed{b} \otimes \boxed{c}$

$\mathcal{B} \otimes \mathcal{B} \otimes \mathcal{B}$ $k = 3$ 27 ELTS.

THEY CAN BE DIVIDED UP AS FOLLOWS:

$a \leq b \leq c$ $\boxed{a} \otimes \boxed{b} \otimes \boxed{c}$

$$\downarrow$$

$$\boxed{a \mid b \mid c}$$

1 1 1
1 1 2
1 1 3
1 2 2
1 2 3
1 3 3
2 2 2
2 2 3
2 3 3
3 3 3

$\mathcal{B}_{(3)}$ = CRYSTAL OF $\mathrm{SYM}^3(\mathbb{C}^3)$ GL(3) REP'N.

IF $a > b > c$ $\boxed{3} \otimes \boxed{2} \otimes \boxed{1}$

$\mathcal{B}_{(111)} = \mathcal{B}_{(1^3)}$ CRYSTAL OF $\Lambda^3(\mathbb{C}^3)$

$\mathcal{B}\otimes\mathcal{B}\otimes\mathcal{B}$ DIVIDES UP.

$\mathcal{B}_{(3)}$ 10

$\mathcal{B}_{(1^3)}$ 1

$\mathcal{B}_{(21)}$ 8 2 COPIES

$$10 + 1 + 2 \cdot 8 = 27.$$

AS $S_3 \times GL(3,\mathbb{C})$ MODULE

$$\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3 =$$

$$\underbrace{1_{S_3}}_{\pi^{S_3}_{(3)}} \otimes \mathrm{SYM}^3(\mathbb{C}^3) \oplus \underbrace{\mathrm{SGN}}_{\pi^{S_3}_{(1^3)}} \otimes \Lambda^3(\mathbb{C}^3)$$

$$\oplus\ \pi^{S_3}_{(2,1)} \otimes \left(\text{ADJOINT SQUARE}\right).$$

2-DIM'L IRREP OF S_3

ASSOCIATIVITY OF TENSOR PRODUCT OF CRYSTALS

THM IF $\mathcal{B}, \mathcal{C}, \mathcal{D}$ ARE GL(n) CRYSTALS

$$(\mathcal{B}\otimes\mathcal{C})\otimes\mathcal{D} \cong \mathcal{B}\otimes(\mathcal{C}\otimes\mathcal{D})$$

$$(b\otimes c)\otimes d \mapsto b\otimes(c\otimes d)$$

IT IS NECESSARY TO CHECK THESE MAPS ARE COMPATIBLE WITH e_i, f_i.

PROOF: FOR EITHER $(\mathcal{B}\otimes\mathcal{C})\otimes\mathcal{D}$ OR $\mathcal{B}\otimes(\mathcal{C}\otimes\mathcal{D})$ THE e_i, f_i HAVE EXPLICIT DESCRIPTION. FOR f_i IT IS

$$f_i(x\otimes y\otimes z) = \begin{cases} f_i(x)\otimes y\otimes z & \varphi_i(x) \\ x\otimes f_i(y)\otimes z & \varphi_i(y)+\varphi_i(x)-\varepsilon_i(x) \\ x\otimes y\otimes f_i(z) & \varphi_i(z)+\varphi_i(y)+\varphi_i(x)-\varepsilon_i(y)-\varepsilon_i(x) \end{cases}$$

WHICH ONE WE CHOOSE IS WHERE THE NUMBER IN SECOND COLUMN IS MAXIMUM. IF THERE ARE 2 PLACES WE CHOOSE THE FIRST.

ONE EXPLICIT WAY OF EMBEDDING $\mathcal{B}_\lambda$ INTO $\otimes^n \mathcal{B}$.

$T \in \mathcal{B}_\lambda$

T A TABLEAU

WRITE OUT ROWS:

$$T = \begin{pmatrix} T_1 \\ T_2 \\ \vdots \\ T_r \end{pmatrix}$$

$T_i \in \mathcal{B}_{(\lambda_i)}$ "CRYSTAL OF ROWS".

TAKE TENSOR PRODUCT IN REVERSE ORDER

$$T = \begin{array}{cccccc} 1 & 1 & 2 & 2 & 3 & 4 \\ 2 & 3 & 3 & 4 & & \\ 3 & & & & & \end{array}$$

$T_1 = \boxed{1} \otimes \boxed{1} \otimes \boxed{2} \otimes \boxed{2} \otimes \boxed{3} \otimes \boxed{4}$

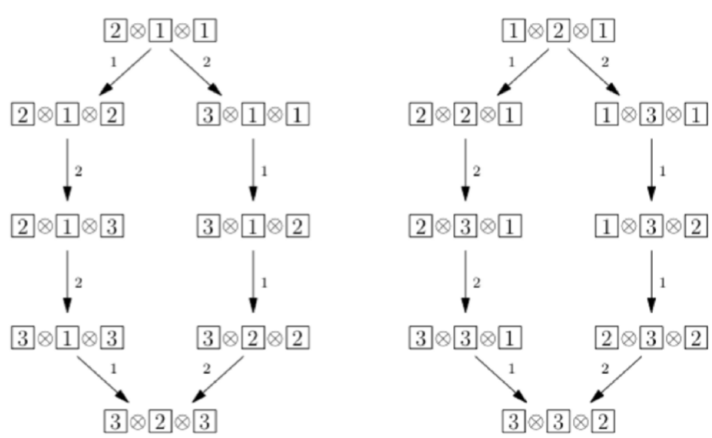
$T_2 = \boxed{2} \otimes \boxed{3} \otimes \boxed{3}$

$T_3 = \boxed{3}$

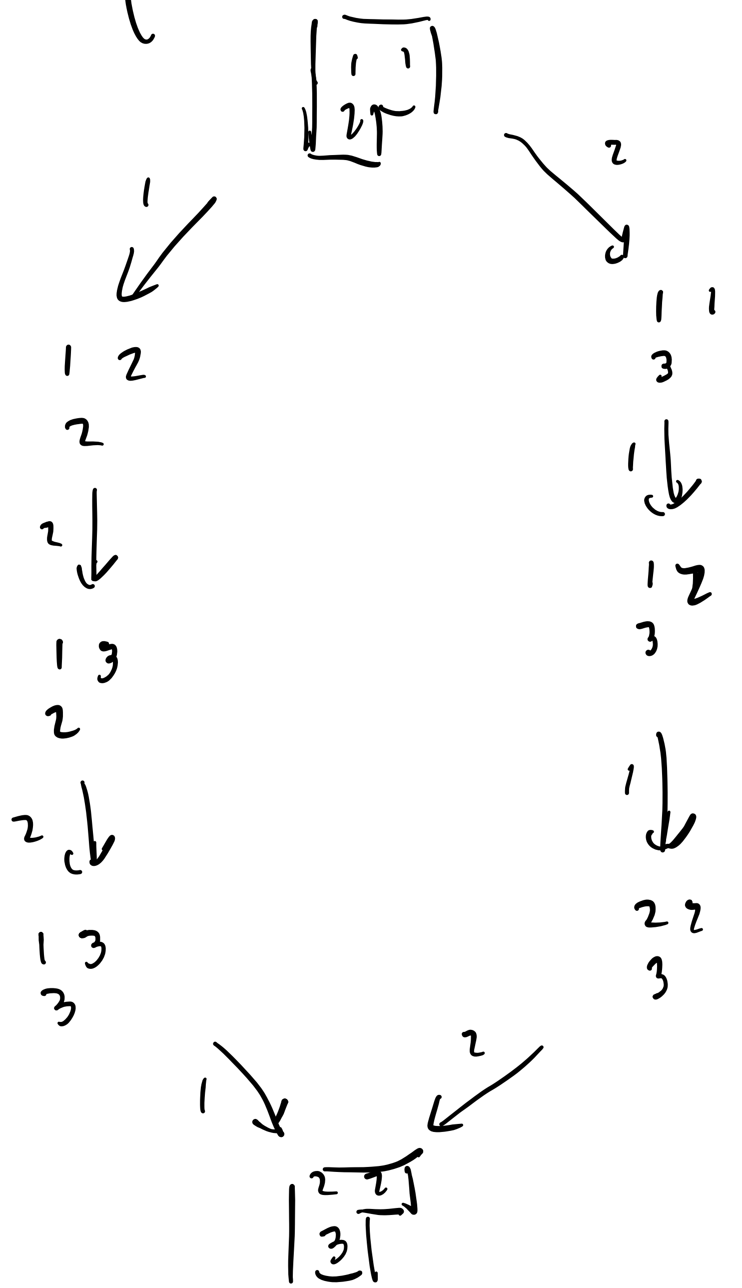
$T \to T_3 \otimes T_2 \otimes T_1$:

$$3 \otimes 2 \otimes 3 \otimes 3 \otimes 1 \otimes 1 \otimes 2 \otimes 2 \otimes 3 \otimes 4.$$

REMARKABLY, IF λ IS FIXED THE SUBSET OF $\otimes^n \mathcal{B}$ OBTAINED THIS WAY IS CLOSED UNDER e_i, f_i. THIS IS ONE REALIZATION OF $\mathcal{B}_\lambda$.



FIRST EMBEDDING.



$\begin{array}{cc}1&1\\2&\end{array} \to \boxed{2}\otimes\boxed{1}\otimes\boxed{1}$

IF $\boxed{a_1} \otimes \boxed{a_2} \otimes \cdots \otimes \boxed{a_n}$

WE CAN PERFORM RSK.

TO DO SCHENSTED INSERTION

$$\emptyset \leftarrow a_1 \leftarrow a_2 \leftarrow \cdots$$

T_1 = SSYT

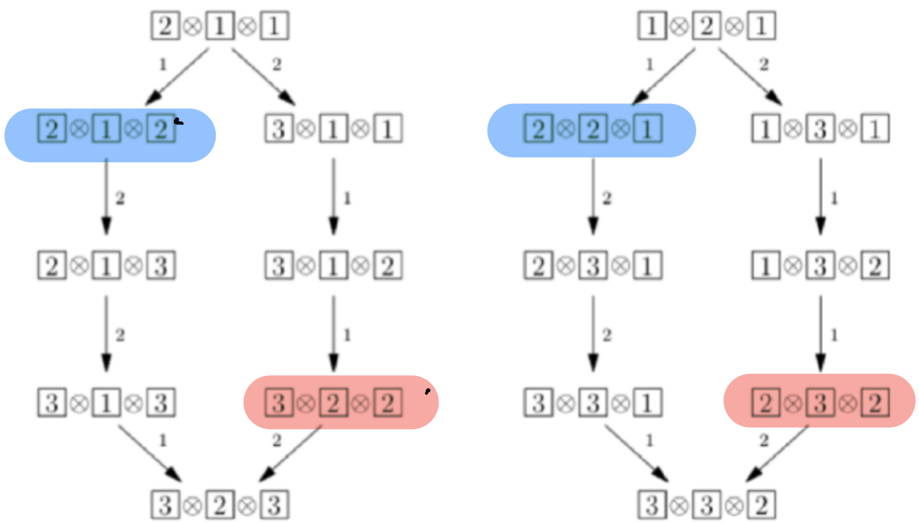
T_2 = STANDARD TABLEAU.

HAVE SAME SHAPE λ.

IF WE FIX A STANDARD TABLEAU T OF SHAPE λ AND COLLECT ALL WORDS $a_1 \cdots a_k$ AS ABOVE SUCH THAT $\underset{RSK}{\rightsquigarrow} (T_1, T_2)$ TABLEAU $T_2 = T$.

THESE FORM A SUBCRYSTAL OF $\otimes^k \mathcal{B}$ ISOMORPHIC TO $\mathcal{B}_\lambda$.

T_1 IS THE ELT OF $\mathcal{B}_\lambda$ CORRESPONDING TO THIS ELT UNDER THIS ISOMORPHISM.



RSK TO 2 1 2.

$$\boxed{2} \qquad \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array} \qquad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array}$$

$T_1 = \begin{matrix} 1 & 2 \\ 2 & \end{matrix}$

$T_2 : \begin{matrix} 1 & 3 \\ 2 & \end{matrix}$

RECORDING TABLEAU.

RSK TO 2 2 1

$$\boxed{2} \qquad \begin{array}{|c|c|}\hline 2 & 2 \\ \hline\end{array} \qquad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array}$$

$$(T_1, T_2) = \begin{array}{cc} 1 & 2 \\ 2 & \end{array} \qquad \begin{array}{cc} 1 & 2 \\ 3 & \end{array}$$

$3 \otimes 2 \otimes 2$

$$\boxed{3} \qquad \begin{array}{|c|}\hline 2 \\ \hline 3 \\ \hline\end{array} \qquad \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}$$

RE CODING $\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}$

HOW CAN THIS BE PROVED ?

PROVE

$$\mathcal{B}(r) \otimes \mathcal{B} \cong \mathcal{B}(r+1) \oplus \mathcal{B}(r,1)$$

CRYSTAL OF ROWS

EXPLICIT

CONSIDER $a \otimes b$

$a = \boxed{a_1} \cdots \boxed{a_r} \in \mathcal{B}(r)$

$a_1 \leq a_2 \leq \cdots \leq a_r$

$b = \boxed{b}$

WHERE DOES $a \otimes b$ APPEAR IN RHS. IF $b \geq a_r$ THEN

$a \otimes b$ APPEARS IN $\mathcal{B}(r+1)$

AS $\boxed{a_1} \otimes \boxed{a_2} \otimes \cdots \otimes \boxed{a_r} \otimes \boxed{b}$.

IF $b < a_r$ THERE IS A

CORRESPONDING ELEMENT OF

$$\mathcal{B}(r,1) = \left\{ \begin{array}{l} \boxed{c_1} \cdots \boxed{c_r} \\ \boxed{d} \end{array} \right\}$$

$c_1 \leq c_2 \leq \cdots \leq c_r \qquad c_1 < d$

EMBEDDED IN $\otimes^{r+1} \mathcal{B}$ AS

$\boxed{d} \otimes \boxed{c_1} \otimes \cdots \otimes \boxed{c_r}$.

WHAT MUST BE CHECKED IF

WE PERFORM SCHENSTED INSERTION

$$\boxed{a_1} \cdots \boxed{a_r} \leftsquigarrow \boxed{b} \qquad b < a_r$$

$$\downarrow$$

$$\begin{array}{|c|c|c|}\hline c_1 & \cdots & c_r \\ \hline d \\ \cline{1-1}\end{array}$$

MAPPING $a \otimes b$ TO THIS

IS COMPATIBLE WITH THE CRYSTAL OPERATIONS e_i, f_i.

IF THIS ISOMORPHISM IS KNOWN, MUCH OF THE RSK DESCRIPTION GIVEN ABOVE OF

$$\bigotimes^n \mathcal{B} \rightsquigarrow \text{SUBCRYSTALS}$$

$$\boxed{a_1} \otimes \cdots \otimes \boxed{a_r} \underset{RSK}{\rightsquigarrow} T_1 \in \mathcal{B}_\lambda.$$

(T_2 TELL US WHICH SUBCRYSTAL.)

$$\mathcal{B}_\lambda = \left\{ \boxed{R_r} \otimes \cdots \otimes \boxed{R_1} \right\}$$

PLACTIC EQUIVALENCE (KNUTH, LASCOUX - SCHÜTZENBERGER)

TWO WORDS

$a_1 \cdots a_k$

$b_1 \cdots b_k$

ARE PLACTICALLY EQUIVALENT IF

$\exists$ ISOMORPHIC SUBCRYSTALS

$\mathcal{C}$ AND $\mathcal{D}$ OF $\otimes^k \mathcal{B}$

$$a = \boxed{a_1} \otimes \cdots \otimes \boxed{a_k} \in \mathcal{C}$$

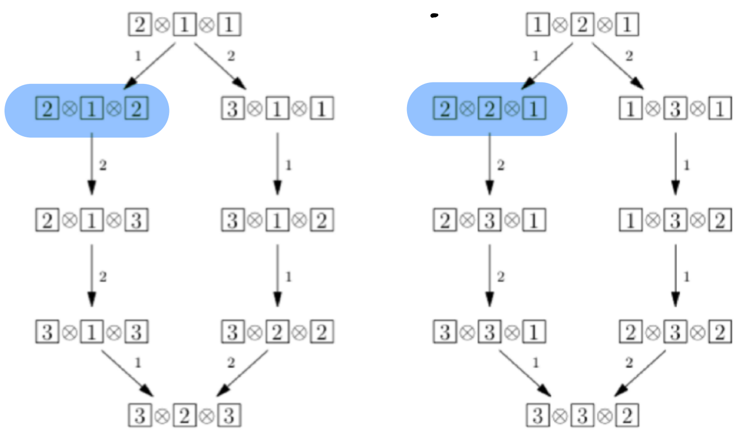
$$b = \boxed{b_1} \otimes \cdots \otimes \boxed{b_k} \in \mathcal{D}$$

AND THE CRYSTAL ISOMORPHISM

$\mathcal{C} \to \mathcal{D}$ TAKES $a \to b$

FOR EXAMPLE

$$\mathcal{C} \longrightarrow \mathcal{D}$$



$$2 \otimes 1 \otimes 2 \to \boxed{2} \otimes \boxed{2} \otimes \boxed{1}$$

$$212 \equiv 221$$

THIS EQUIVALENCE IS SAME AS KNUTH EQUIVALENCE THOUGH DEFINITION WOULD BE SLIGHTLY DIFFERENT.

$$\mathcal{B}_{(r)} \otimes \mathcal{B} \cong \mathcal{B}_{(r+1)} \oplus \mathcal{B}_{(r,1)}$$

$$\boxed{a} \quad \boxed{b} \quad \rightsquigarrow \quad \begin{array}{|c|}\hline c \\ \hline d \\ \hline \end{array}$$

$$\boxed{2\ 2} \otimes \boxed{1} \rightsquigarrow \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 & \\ \cline{1-1}\end{array}$$

$\in \mathcal{B}_{(2)}$

$$= \boxed{2} \otimes \boxed{1} \otimes \boxed{2}.$$

$221 \equiv 212$ PLACTICALLY EQUIV.

FOR MORE COMPLICATED WORDS.

$$\mathcal{B}_\lambda = \{T_r \otimes T_{r-1} \otimes \cdots \otimes T_1\}$$

T_1
T_2
$\vdots$
T_r IS A SST TABLEAU

REPEATEDLY APPLY THIS ISOMORPHISM WILL SHOW ANY WORD IS P.E. TO ONE OF THESE.

NEXT WEEK:
PLACTIC AND KNUTH EQUIVALENCE

MORE DETAILS ABOUT RSK - CRYSTAL CONNECTION.

$$T \rightsquigarrow \otimes^{n} \mathbb{B}$$

$$T_r \otimes \cdots \otimes T_1 \otimes \mathbb{B}$$

$$T_1 \otimes \mathbb{B} \equiv T_1' \in \mathbb{B}_{(r+1)}$$

$$\text{or } \boxed{\lambda} \otimes T_1' \qquad T_1' \in \mathbb{B}_{(r)}$$

$$\in \mathbb{B}_{(r,1)}$$

REPEAT. $(r = \lambda_1)$